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# ON THE TIME-DEPENDENT CONVECTIVE HEAT TRANSFER IN FLUIDS WITH VANISHING PRANDTL NUMBER†

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**Abstract**—Analytical solutions are presented for the transient heat transfer in forced convection flow over a curved wall when the Prandtl number is zero. The wall is subjected to either a variation in temperature or a variation in the heat flux. As an application, the solutions are computed for the case of wedge flows. The present analysis also should be applicable to a laminar boundary layer flow with a variable free stream velocity when the Prandtl number is slightly different from zero.

## NOMENCLATURE

$erf(x)$, error function $= \frac{2}{\sqrt{\pi}} \int_0^x \exp.(-\theta^2)\, d\theta$;

$F(t)$, defined in equation (27);

$f(x)$, dimensionless function related to the wall temperature, see equation (8);

$K$, thermal conductivity;

$L$, characteristic length;

$m$, exponent, in connection with the potential velocity component in wedge flows;

$N(x, y; \eta)$, function defined in equation (41);

$Pe = \frac{UL}{\kappa}$, Péclet number;

$q_0$, constant heat flux along the wall;

$Re = \frac{UL}{\nu}$, Reynolds number;

$T(t, x, y)$, fluid temperature change over the temperature far from the wall;

$T_w$, constant temperature along the wall;

$t$, dimensionless time;

$U$, reference velocity;

$u(x, y)$, $v(x, y)$, dimensionless fluid velocity components;

$u_p(x, y)$, $v_p(x, y)$, dimensionless fluid potential velocity components;

$X(x) = \int_0^x \frac{d\eta}{u_p(\eta)}$, dimensionless distance along streamline;

$x$, dimensionless distance along the wall;

$Y_1(t, x, \eta)$, $Y_2(t, x, \eta)$, functions defined in equation (19);

$y$, dimensionless distance perpendicular to the wall;

$\alpha$, variable, defined in equation (47);

$B_\alpha(n, m)$, incomplete Beta function, defined in equation (51);

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$\Gamma(x)$, Gamma function $= \int_0^\infty \exp.(-\zeta) \times \zeta^{x-1}\, d\zeta$;
$\gamma(x)$, dimensionless function related to wall heat flux, see equation (9);
$\delta$, thickness of the hydrodynamic boundary layer;
$\zeta = \sqrt{(Pe)}\, y$, dimensionless distance normal to the wall;
$\theta$, Fourier sine or cosine transform of $T(t, x, \psi)$ with respect to the $\psi$ variable;
$\kappa$, thermal diffusivity;
$\nu$, kinematic viscosity;
$\sigma = \nu/\kappa$, Prandtl number;
$\tau = tu_p(x)/x$, dimensionless time variable;
$\tau_c = t_1 U/x_1$, dimensionless time variable in [3] and [4];
$\phi$, function defined in equation (26);
$\chi$, arbitrary function;
$\psi(x, \zeta)$, stream function.

Subscripts
1, dimensional quantities;
$q$, the case of a prescribed wall heat flux;
$ss$, steady-state conditions;
$w$, conditions at the wall.

Superscript
$'$, differentiation.

## INTRODUCTION

THE PROBLEM of the heat transfer to fluids of very small Prandtl numbers (for example, liquid metals) is of considerable practical importance in nuclear-power plants and in power generation systems for space applications. The problem has been analyzed frequently in the literature.

In steady-state heat transfer, Morgan *et al.* [1] presented a theoretical study for incompressible laminar boundary layer flows around bodies of arbitrary shape when the Prandtl number $\sigma$ is very small. The analysis was based on the observation that for small $\sigma$ the hydrodynamic boundary layer is thin compared to the thermal boundary layer so that the velocity distribution is essentially potential over the major portion of the thermal layer. A solution was obtained for the first two terms of a series representation for the temperature in powers of $\sigma^{\frac{1}{2}}$ for variable wall temperature. Napolitano [2] treated the same problem, but for the special case of a flat plate at constant wall temperature. His final result, for the wall heat flux, was also in the form of a series in powers of $\sigma^{\frac{1}{2}}$. The first three terms were determined and the first two terms were identical to those of Morgan [1]. Napolitano concluded that the accuracy of his solution is satisfactory even for a Prandtl number as high as 0·6.

The problem of transient heat transfer when the Prandtl number is small was treated by Cess [3] for the case of an incompressible laminar boundary layer flow over a flat plate with a step in the wall temperature. Cess used the observation of Reference [1] and approximated the velocity field throughout the thermal layer by its value at the edge of the hydrodynamic boundary layer. He obtained a solution for the wall heat flux for small and large times and these were joined at $\tau_c = t_1 U/x_1 = 1$. The solution for small times was in the form of a series in powers of $\tau_c^{\frac{1}{2}}$. In treating the same problem, Riley [4] pointed out that, for very small times, the thermal layer is totally embedded in the velocity boundary layer. He used the series expansion of the Blasius velocity function near the wall and found the solution in the form of a series in powers of $\tau_c^{\frac{3}{2}}$. For large times, Riley showed that the steady state is approached in an exponential manner. No matching of the two solutions was attempted.

The present paper treats the problem of transient heat transfer to laminar boundary layer flows when the Prandtl number is zero. This means that the velocity boundary layer vanishes and the velocity distribution can be approximated by the potential flow solution throughout the thermal layer. Analytical solutions are obtained for arbitrary potential flows when the wall is subjected to a variation in temperature or heat flux.

## ANALYSIS

The starting point in the analysis is the transient form of the energy equation for plane incompressible laminar boundary layer flow.

$$\frac{\partial T_1}{\partial t_1} + u_1(x_1, y_1)\frac{\partial T_1}{\partial x_1} + v_1(x_1, y_1)\frac{\partial T_1}{\partial y_1} = \kappa\frac{\partial^2 T_1}{\partial y_1^2} \tag{1}$$

where $x_1$ and $y_1$ are the coordinates along and perpendicular to the possibly curved wall, respectively, see Fig. 1. In the energy equation, the

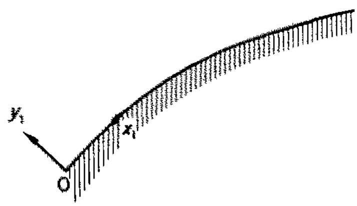


FIG. 1. The coordinate system.

conduction in the $x_1$ direction as well as the viscous dissipation in the fluid have been neglected and the fluid properties are taken to be constant. The velocity field is assumed to be independent of time.

It is well known that the velocity boundary layer thickness is $\delta\alpha\sqrt{v}$. Hence, when $v \to 0$, $\sigma = v/\kappa \to 0$ and $\delta \to 0$ and hence the flow is potential throughout the thermal layer. Consequently, the velocity components $u_1$ and $v_1$ reduce to the potential flow components $u_{p1}$ and $v_{p1}$. If the Péclet number $Pe = UL/\kappa \gg 1$, the thickness of the thermal layer is very small compared to the characteristic geometric dimension of the problem $L$ [5], and hence the potential velocity components vary only slightly across the thickness of the thermal layer. If the components $u_{p1}$ and $v_{p1}$ are expanded in power series in the variable $y_1$ about $y_1 = 0$ (see also [1]), and only the first term in these series is retained, the velocity components $u_1$ and $v_1$ take on the form

$$u_1(x_1, y_1) = u_{p1}(x_1) \tag{2}$$

$$v_1(x_1, y_1) = v_{p1}(x_1, y_1) = -y_1 u'_{p1}(x_1). \tag{3}$$

Note that the velocity component $v_{p1}$ is such that $u_{p1}$ and $v_{p1}$ satisfy the continuity equation for incompressible flow. Equation (1) can now be written in a nondimensional form using equations (2) and (3) with the result that

$$\frac{\partial T}{\partial t} + u_p(x)\frac{\partial T}{\partial x} - \zeta u'_p(x)\frac{\partial T}{\partial \zeta} = \frac{\partial^2 T}{\partial \zeta^2}, \quad t > 0, x > 0, \zeta > 0 \tag{4}$$

where

$$t = \frac{t_1 U}{L}, x = \frac{x_1}{L}, y = \frac{y_1}{L}, u_p = \frac{u_{p1}}{U}, T = \frac{T_1}{T_w}$$

and

$$\zeta = \sqrt{(Pe)}\, y. \tag{5}$$

The side conditions are

$$T(0, x, \zeta) = 0, x > 0, \zeta > 0; T(t, 0, \zeta) = 0, \quad t > 0, \zeta > 0 \tag{6}$$

$$T(t, x, \infty) = 0, t > 0, x > 0 \tag{7}$$

and

$$T(t, x, 0) = f(x), t > 0, x > 0 \tag{8}$$

or

$$-\frac{\partial Tq}{\partial \zeta}(t, x, 0) = \gamma(x), \quad t > 0, x > 0 \left[T_q = \frac{T_1 K\sqrt{Pe}}{q_0 L}\right] \tag{9}$$

where the functions $f(x)$ and $\gamma(x)$ are prescribed functions of $x$ which are specified later in the analysis. The boundary conditions expressed in equations (8) and (9) describe either a variation in the wall temperature [equation (8)] or a variation in the wall heat flux [equation (9)]. Note that $T_w$ and $q_0$ represent constant dimensional temperature and heat flux along the wall.

According to conditions in equations (6) and (7), the fluid temperature is initially uniform throughout the space and retain this value for $t > 0$, at the leading edge of the surface and far from the wall.

The system of equations (4)–(9) can be simplified by applying the von Mises transformation. The stream function $\psi$ is defined as

$$\psi = u_p(x)\,\zeta = \sqrt{(Pe)}\,u_p(x)\,y. \tag{10}$$

With this, equations (4)–(9) change to

$$\frac{\partial T}{\partial t} + u_p(x)\frac{\partial T}{\partial x} = [u_p(x)]^2\frac{\partial^2 T}{\partial \psi^2}, \qquad t > 0, x > 0, \psi > 0 \tag{11}$$

$$T(0, x, \psi) = 0, x > 0, \psi > 0; \qquad T(t, 0, \psi) = 0, t > 0, \psi > 0 \tag{12}$$

$$T(t, x, \infty) = 0, t > 0, x > 0 \tag{13}$$

and

$$T(t, x, 0) = f(x), t > 0, x > 0 \tag{14}$$

or

$$-\frac{\partial T_q}{\partial \psi}(t, x, 0) = \frac{\gamma(x)}{u_p(x)}, t > 0, x > 0. \tag{15}$$

These equations are solved by applying a Fourier transformation with respect to the variable $\psi$. This results in first order partial differential equations, with $t$ and $x$ as independent variables, which are solved by the method of characteristics. This method of solution can be also used in case $f$, $\gamma$, and $u_p$ are functions of both $x$ and $t$. The equations, however, become more involved and are not considered in the present paper. The case of a variation in wall temperature will be treated first.

*Solution for the heat flux when the wall temperature is prescribed*

The problem in this case is described by equations (11)–(14). The solution is obtained by first applying a Fourier sine transform to equation (11) with respect to the variable $\psi$. The Fourier sine transform of a function $T(t, x, \psi)$ is defined as [6].

$$\theta(t, x, \omega) = \sqrt{(2/\pi)}\int_0^\infty T(t, x, \psi)\sin(\omega\psi)\,\mathrm{d}\psi. \tag{16}$$

Using equation (16) and the boundary conditions (13) and (14), equation (11) transforms into the first order partial differential equation

$$\frac{\partial \theta}{\partial t} + u_p(x)\frac{\partial \theta}{\partial x} = u_p^2(x)\,[\sqrt{(2/\pi)}\,\omega f(x) - \omega^2\theta]. \tag{17}$$

Equation (17) can be solved by the method of characteristics subject to the side conditions in equation (12), which when transformed read,

$$\theta(0, x, \omega) = 0, \theta(t, 0, \omega) = 0. \tag{18}$$

As described in [7], the general solution of equation (17) is of the form

$$Y_2 = \chi(Y_1) \tag{19}$$

where $Y_1(t, x, \theta) = C_1$ and $Y_2(t, x, \theta) = C_2$ are solutions of any two independent differential equations that imply the relationship

$$\mathrm{d}t = \frac{\mathrm{d}x}{u_p(x)} = \frac{\mathrm{d}\theta}{u_p^2(x)\,[\sqrt{(2\pi)}\,\omega f(x) - \omega^2\theta]}. \tag{20}$$

In general, the solutions $Y_1$ and $Y_2$ represent two families of integral surfaces, $Y_1(t, x, \theta) = C_1$ and $Y_2(t, x, \theta) = C_2$ in the coordinate system $t, x, \theta$. The intersection of a surface of one family with a surface of the second family defines a characteristic curve. $Y_1$ and $Y_2$ are determined by integrating the following two equations:

$$\mathrm{d}t = \frac{\mathrm{d}x}{u_p(x)} \tag{21}$$

$$u_p(x)\,\mathrm{d}x = \frac{\mathrm{d}\theta}{[\sqrt{(2\pi)}\,\omega f(x) - \omega^2\theta]}. \tag{22}$$

The solution to equation (21) is

$$Y_1(t, x, \theta) = t - X(x) = C_1 \tag{23}$$

where

$$X(x) = \int_0^x \frac{d\eta}{u_p(\eta)}, x \geqslant 0. \tag{24}$$

The potential velocity distribution should be of such character that the integral in equation (24) exists. After integrating both sides of equation (22), and rearranging the terms, the following is obtained:

$$Y_2(t, x, \theta) \equiv \sqrt{(2/\pi)}\frac{f(x)}{\omega}\exp[\omega^2\phi(x)] - \theta \exp.[\omega^2\phi(x)] - \frac{\sqrt{(2/\pi)}}{\omega}\int_0^x f'(\eta)\exp[\omega^2\phi(\eta)]\,d\eta = C_2 \tag{25}$$

where

$$\phi(x) = \int_0^x u_p(\eta)\,d\eta, x \geqslant 0. \tag{26}$$

In this case, the integral surface $Y_1 = C_1$ is a family of planes that are parallel to the $x$–$t$ plane. The equation $Y_2 = C_2$ defines a family of surfaces since $\theta$ is a function of $x$ and $t$. Any plane of the first family intersects a surface of the second family in a characteristic curve that lies in a plane parallel to the $x$–$t$ plane where our interest is confined to the first quadrant $x > 0$, $t > 0$. These characteristic curves are readily defined by equation (23) and are shown in Fig. 2. The equation of the dividing characteristic, which passes through the origin in the $x$–$t$ plane, is defined as

$$t = X(x) \text{ or } x = F(t) \tag{27}$$

on the condition that equation (23) can be inverted and there is a 1–1 correspondence between the variables $t$ and $x$. Except for an arbitrary constant, the equation of the other characteristic curves shown in Fig. 2 is the same as equation (27) and it may be written either as in equation (23) or as

$$x - F(t) = C_3. \tag{28}$$

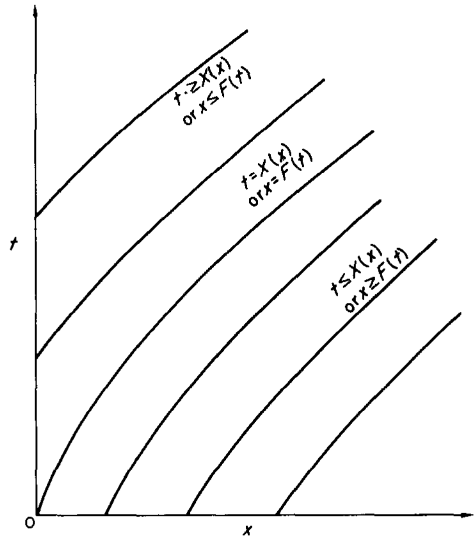


FIG. 2. Characteristic curves in the $x$–$t$ plane ($x > 0$, $t > 0$).

Hence, in view of equations (19), (25) and (28) the general solution of the partial differential equation (17) is

$$\theta(t, x, \omega) = \sqrt{(2/\pi)}\frac{f(x)}{\omega} - \frac{\sqrt{(2/\pi)}}{\omega} \times \exp[-\omega^2\phi(x)]\int_0^x f'(\eta)\exp[\omega^2\phi(\eta)]\,d\eta - \chi[x - F(t)]\exp.[-\omega^2\phi(x)]. \tag{29}$$

The arbitrary function $\chi$ is determined such that the initial conditions for $\theta$ as stipulated in equation (18) are satisfied. This results in two different expressions for $\chi$ and hence for $\theta$ depending on whether $t \leqslant X(x)$ or $t \geqslant X(x)$

$$\theta(t, x, \omega) = \frac{\sqrt{(2/\pi)}}{\omega}f(x) - \frac{\sqrt{(2/\pi)}}{\omega}f[x - F(t)] \times \exp[-\omega^2\{\phi(x) - \phi(x - F(t))\}] - \frac{\sqrt{(2/\pi)}}{\omega}\int_{x-F(t)}^{x} f'(\eta)\exp[-\omega^2\{\phi(x) - \phi(\eta)\}]\,d\eta, t \leqslant X(x) \tag{30}$$

$$\theta(t, x, \omega) = \frac{\sqrt{(2/\pi)}}{\omega} f(x) - \frac{\sqrt{(2/\pi)}}{\omega} f(0) \times \exp[-\omega^2 \phi(x)] - \frac{\sqrt{(2/\pi)}}{\omega} \int_0^x f'(\eta) \times \exp[-\omega^2\{\phi(x) - \phi(\eta)\}] d\eta, t \geqslant X(x). \tag{31}$$

Note that the wall temperature $f(x)$ can have a finite step $f(0)$ at $x = 0$. In equation (30), $x - F(t) > 0$ (see Fig. 2) until $t = X(x)$, that is, $x = F(t)$ when $f[x - F(t)] = f(0)$. Furthermore the function $\phi(x)$ is defined in equation (26) for $x \geqslant 0$ and hence $\phi[x - F(t)]$ in equation (30) is well defined for $x - F(t) \geqslant 0$. Equation (30) satisfies the condition $\theta(0, x, \omega) = 0$ and corresponds to the transient part of the solution. Similarly, equation (31) satisfies the condition $\theta(t, 0, \omega) = 0$ and represents the steady-state part of the solution.

The next step is to invert equations (30) and (31) back to the $(t, x, \psi)$ domain. This is done by using the relationships [6]

$$\int_0^\infty \frac{\sin(\omega\psi)}{\omega} d\omega = \frac{\pi}{2}, \psi > 0 \tag{32}$$

$$\int_0^\infty \frac{\exp(-\omega^2 a)}{\omega} \sin(\omega\psi) d\omega = \operatorname{erf}(\psi/2\sqrt{a}), a \geqslant 0, \psi > 0 \tag{33}$$

where $a$ is a parameter, and the assumption is made that the integrations in equation (33) can be interchanged with the integrals appearing in equations (30) and (31). Using the above relationships and equation (10), the final result of the fluid temperature distribution in the dimensionless $(t, x, y)$ domain is obtained as

$$T(t, x, y) = f(x) - f[x - F(t)] \times \operatorname{erf}[\sqrt{(Pe)}\, u_p(x)\, y/2\, \{\phi(x) - \phi[x - F(t)]\}^{\frac{1}{2}}] - \int_{x-F(t)}^{x} f'(\eta) \operatorname{erf}\{\sqrt{(Pe)}\, u_p(x)\, y/2\, [\phi(x) - \phi(\eta)]^{\frac{1}{2}}\} d\eta, t \leqslant X(x) \tag{34}$$

$$T(t, x, y) = f(x) - f(0) \times \operatorname{erf}\{\sqrt{(Pe)}\, u_p(x)\, y/2\, [\phi(x)]^{\frac{1}{2}}\} - \int_0^x f'(\eta) \operatorname{erf}\{\sqrt{(Pe)}\, u_p(x)\, y/2\, [\phi(x) - \phi(\eta)]^{\frac{1}{2}}\} d\eta, t \geqslant X(x). \tag{35}$$

These expressions can be verified and shown to satisfy the partial differential equation and the associated side conditions.

Our primary interest centers on the dimensionless temperature gradient at the wall. This is obtained by differentiating equations (34) and (35) with respect to $y$ and then letting $y$ tend to zero.

$$-\frac{\partial T}{\partial y}(t, x, 0) = \frac{\sqrt{(Pe)}\, u_p(x) f[x - F(t)]}{\sqrt{(\pi)}\, [\phi(x) - \phi\{x - F(t)\}]^{\frac{1}{2}}} + \frac{\sqrt{(Pe)}\, u_p(x)}{\sqrt{(\pi)}} \int_{x-F(t)}^{x} \frac{f'(\eta)\, d\eta}{[\phi(x) - \phi(\eta)]^{\frac{1}{2}}}, \quad t \leqslant X(x) \tag{36}$$

$$-\frac{\partial T}{\partial y}(t, x, 0) = \frac{\sqrt{(Pe)}\, u_p(x) f(0)}{\sqrt{(\pi)}\, [\phi(x)]^{\frac{1}{2}}} + \frac{\sqrt{(Pe)}\, u_p(x)}{\sqrt{\pi}} \times \int_0^x \frac{f'(\eta)\, d\eta}{[\phi(x) - \phi(\eta)]^{\frac{1}{2}}}, t \geqslant X(x). \tag{37}$$

It is important here to note that in case $f(x)$ has a step jump at $x = 0$ and remains constant thereafter the integrals in equations (36) and (37) vanish. The integrals in equations (36) and (37) exist as $\eta$ tends to $x$ provided $f(\eta)$ and $\phi(\eta)$ are such that

$$\frac{f'(\eta)}{[\phi(x) - \phi(\eta)]^{\frac{1}{2}}} = 0(\eta^{-n}), n < 1$$

in the neighborhood of the singular point of the integrand.

Equations (36) and (37) reveal that for time $t < X(x)$, the temperature gradient at the wall is a function of both time and location along the wall and for $t \geqslant X(x)$, the temperature gradient at the wall reaches steady-state conditions and hence is independent of time.

*Solution for the wall temperature when the heat flux is prescribed*

The problem in this case is described in equations (11), (12), (13) and (15). The analysis follows closely that presented in the previous section. The only major difference is the use of the Fourier cosine transform, which is defined as [6]

$$\theta(t, x, \omega) = \sqrt{(2/\pi)} \int_0^\infty T_q(t, x, \psi) \,.\, \cos(\omega\psi)\, d\psi \tag{38}$$

instead of the Fourier sine transform, which was previously employed. The rest of the details are omitted and the final results for the dimensionless temperature distribution are

$$T_q(t, x, y) = \int_{x-F(t)}^{x} N(x, y; \eta)\, d\eta,\ t \leqslant X(x) \tag{39}$$

$$T_q(t, x, y) = \int_0^x N(x, y; \eta)\, d\eta,\ t \geqslant X(x) \tag{40}$$

where

$$N(x, y; \eta) = \frac{\gamma(\eta)}{\sqrt{(\pi)}\,[\phi(x) - \phi(\eta)]^{\frac{1}{2}}} \times \exp\left\{-\frac{Pe u_p^2(x)\, y^2}{4[\phi(x) - \phi(\eta)]}\right\}. \tag{41}$$

By substituting $y = 0$, the wall temperature can be easily obtained from equations (39) and (40). The integrals in the above equations converge as $\eta$ tends to $x$ provided $\gamma(\eta)$ and $\phi(\eta)$ are such that

$$\frac{\gamma(\eta)}{[\phi(x) - \phi(\eta)]^{\frac{1}{2}}} = 0(\eta^{-n}), n < 1$$

in the neighborhood of the singular point of the integrand.

## APPLICATIONS

To demonstrate the applications of the present analysis, the problem of wedge flows will be considered. In this case, the dimensionless potential velocity is [5]

$$u_p(x) = x^m, 0 \leqslant m < 1, x \geqslant 0 \tag{42}$$

where $x$ is a dimensionless distance measured along the surface of the wedge from its vertex. The functions $X(x)$ and $F(t)$, defined in equations (24) and (27) are

$$X(x) = x^{1-m}/(1 - m), x \geqslant 0 \text{ and } F(t) = [t(1 - m)]^{1/(1-m)}, t \geqslant 0 \tag{43}$$

while

$$\phi(x) = x^{m+1}/(m + 1), x \geqslant 0.$$

Note that the case of $m = 1$ (stagnation-point flow) has been excluded since the integral in equation (24) diverges at the lower limit. For simplicity, the prescribed wall temperature or heat flux are assumed to be uniform spatially, hence

$$f(x) = \gamma(x) = 1, x \geqslant 0.$$

*Step jump in wall temperature*

The ratio of the time-dependent heat flux at the wall to its steady-state value ($q_{1,w}/q_{1,w,ss}$) is obtained from equations (36), (37) and (42)–(44).

$$q_{1,w}/q_{1,w,ss} = 1/(1 - \alpha^{m+1})^{\frac{1}{2}}, \tau \leqslant 1/(1 - m) \tag{45}$$

$$q_{1,w}/q_{1,w,ss} = 1, \tau \geqslant 1/(1 - m) \tag{46}$$

where

$$\alpha = \{1 - [\tau(1 - m)]^{1/(1-m)}\}. \tag{47}$$

In this case, $\tau = tu_p(x)/x = tx^{m-1}$ and the steady-state wall heat flux is given by equation (48)

$$q_{1,w,ss} = \frac{KT_w}{L}\left(\frac{Pe(m + 1)}{\pi}\right)^{\frac{1}{2}} \cdot \left(\frac{x_1}{L}\right)^{\frac{m-1}{2}}. \tag{48}$$

To compute $q_{1,w,ss}$, $L$ should be taken as a typical length in the $x$-direction; the reference velocity $U$, appearing in $Pe$, should be considered as the velocity at $x_1 = L$.

The case of a flow over a flat plate corresponds to $m = 0$ and equations (45) and (46) reduce to

$$q_{1,w}/q_{1,w,ss} = 1/\sqrt{\tau}, \tau \leqslant 1 \quad (49)$$

$$q_{1,w}/q_{1,w,ss} = 1, \tau \geqslant 1. \quad (50)$$

These results are shown in Fig. 3 for different values of $m$.

When $\alpha = 1$, the integral in equation (51) defines the complete Beta function, which is related to the Gamma function by the following relationship [8]

$$B\left(\frac{1}{m+1}, \frac{1}{2}\right) = \sqrt{(\pi)}\Gamma\left(\frac{1}{m+1}\right)/\Gamma\left(\frac{m+3}{2m+2}\right) \quad (52)$$

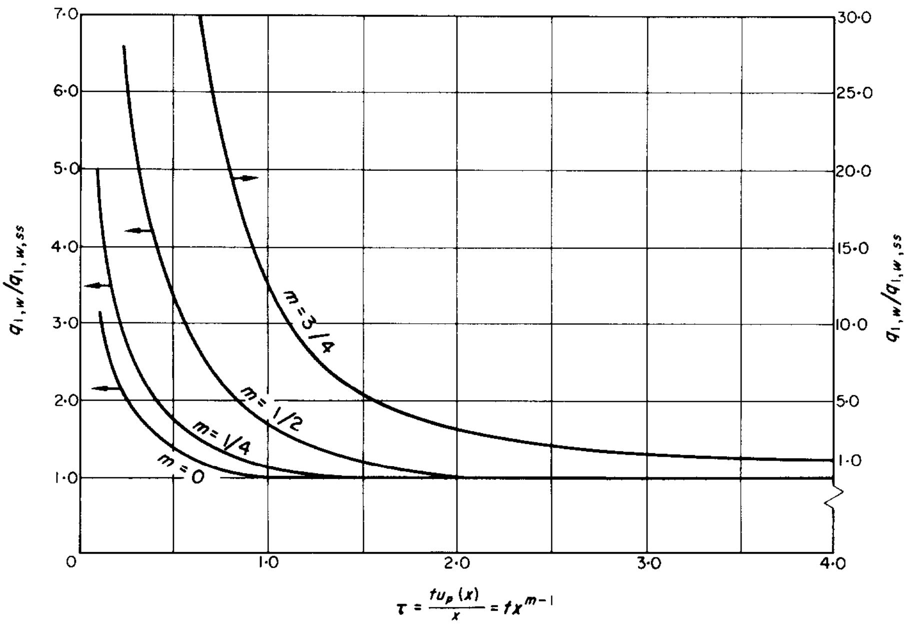


FIG. 3. Transient wall heat flux response to a step jump in wall temperature for wedge flows, equations (45) and (46).

*Step jump in wall heat flux*

The ratio of the time-dependent wall temperature to its steady-state value is calculated from equations (39)–(41) ($y = 0$), (9), (42)–(44). The integrals appearing in equations (39) and (40) can be written, in this case, in terms of the incomplete Beta function $B_\alpha$ $(1/[m + 1], \frac{1}{2})$ which is defined as [8]

$$B_\alpha\left(\frac{1}{m+1}, \frac{1}{2}\right) = \int_0^\alpha \zeta^{-\frac{m}{m+1}} (1 - \zeta)^{-\frac{1}{2}} \, d\zeta. \quad (51)$$

The incomplete Beta function is tabulated in [8]. Using the above relationships, the ratio of the wall temperature to its steady-state value can be written in the following form

$$T_{1,q,w}/T_{1,q,w,ss} = 1 - B_\alpha\left(\frac{1}{m+1}, \frac{1}{2}\right) \times \Gamma\left(\frac{m+3}{2m+2}\right)\Big/\left[\sqrt{(\pi)}\Gamma\left(\frac{1}{m+1}\right)\right] \tau \leqslant \frac{1}{1-m} \quad (53)$$

$$T_{1,q,w}/T_{1,q,w,ss} = 1, \tau \geqslant \frac{1}{1-m}. \qquad (54)$$

The steady-state wall temperature in this case is

$$T_{1,q,w,ss} = \frac{q_0 L}{K[(m+1)\,Pe]^{\frac{1}{2}}} \times \frac{\Gamma\left(\dfrac{1}{m+1}\right)}{\Gamma\left(\dfrac{m+3}{2m+2}\right)} \left(\frac{x_1}{L}\right)^{\frac{1-m}{2}}. \qquad (55)$$

$T_{1,q,w,ss}$ is computed in the same way as $q_{1,w,ss}$, given in equation (48). When $m = 0$ (flat plate)

$$T_{1,q,w}/T_{1,q,w,ss} = \sqrt{(\tau)}, \tau \leqslant 1 \qquad (56)$$

$$T_{1,q,w}/T_{1,q,w,ss} = 1, \tau \geqslant 1. \qquad (57)$$

The above results are shown in Fig. 4 for different values of $m$. Since the arguments of $B_\alpha$ for $m = \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$ are not tabulated in [8], $B_\alpha$ in equation (52) was evaluated on an IBM 360 computer.

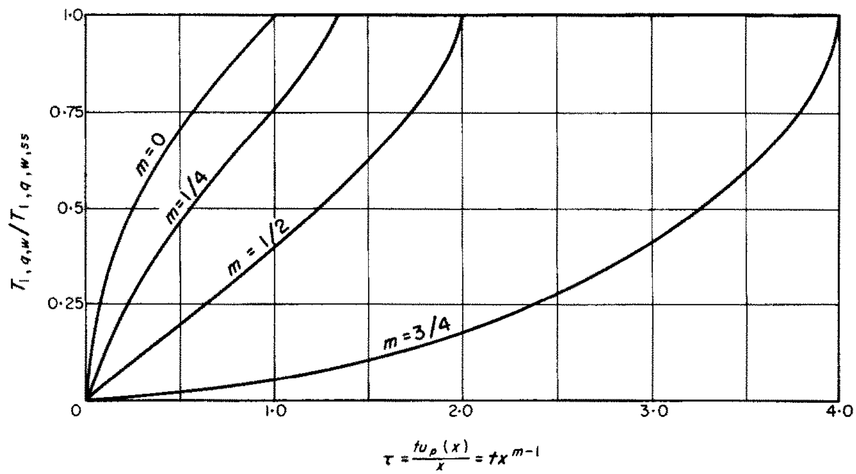


FIG. 4. Transient wall temperature response to a step jump in wall heat flux for wedge flows, equations (53) and (54).

## DISCUSSION

The solutions for both the heat flux when the wall temperature is prescribed constant, and the wall temperature when the heat flux is prescribed constant are shown in Figs. 3 and 4. The values $m = 0, \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$ correspond to the wedge angles 0, $2\pi/5$, $2\pi/3$, and $6\pi/7$, respectively [5, p. 143].

These solutions are composed of a transient part that reaches the steady state in an abrupt manner. This abrupt behavior is a result of the assumption that $\sigma = 0$ and that the potential velocity prevails throughout the thermal layer.

The present analysis, though developed for the theoretical case of $\sigma = 0$ (which implies the absence of a hydrodynamic boundary layer), may still give good results for laminar boundary layers when $\sigma$ is slightly different from zero. (Note that in this case the Péclet number should be written as $Pe = Re \,.\, \sigma$). This is suggested by the results of [1] and [4]. The steady-state part of the present solution for a prescribed wall temperature, that is, equation (37), is identical to the first term in the series expansion of the wall heat flux in [1]. This term was shown [1] to predict the heat transfer from the wall quite well up to $\sigma = 0{\cdot}03$. During the transient phase, the small time solution by Riley [4], for the flat plate case revealed that the heat transfer process is dominated primarily by the one-dimensional molecular diffusion transverse to the flow field.

This is also the case in the present solution, equations (49) and (56). The large time solution [4] consisted of a steady-state part and a transient contribution in the form of a series whose leading term was determined. This term shows that the steady state is reached exponentially and not abruptly at $\tau = 1$ as indicated in equations (50) and (57). However, for small Prandtl numbers and for values of $\tau \cong 1$, this term becomes small when compared to the steady-state part of the solution and therefore the abrupt behavior shown in the present analysis should be a good approximation.

This indicates, at least for the special case of a flat plate, that during transient and steady-state phases, the present analysis should predict the heat transfer or surface temperature quite well if the Prandtl number is sufficiently small. The present analysis is of particular value in case of geometries other than a flat plate where no analytical solutions are available in the literature.

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**Résumé**—Des solutions analytiques sont présentées pour le transport de chaleur transitoire dans un écoulement avec convection forcée sur une paroi courbe lorsque le nombre de Prandtl est nul. La paroi est soumise soit à une variation de température soit à une variation de flux de chaleur. Comme application, on a calculé les solutions pour le cas des écoulements sur des dièdres. L'analyse actuelle serait également applicable à un écoulement laminaire de couche limite avec une vitesse variable de l'écoulement libre lorsque le nombre de Prandtl est légèrement différent de zéro.

**Zusammenfassung**—Für den Wärmeübergang in einer Zwangskonvektionströmung an einer gekrümmten Wand werden analytische Lösungen agegeben, für eine Prandtl-Zahl gleich Null. Für die Wand ist dabei eine Variation der Temperatur oder de Wärmestromdichte vorgegeben. Als Anwendungsbeispiele wurden die Lösungen für Keilströmungen berechnet. Die vorliegende Methode dürfte auch auf laminare Grenzschichtströmungen mit variabler Freistromgeschwindigkeit anwendbar sein, wenn die Prandtl-Zahl nur wenig grösser als Null ist.

**Аннотация**—Представлены аналитические решения нестационарного переноса тепля при вынужденной конвекции на искривленной стенке, когда критерий Прандтля равен нулю. Изменялась или температура стенки или тепловой поток. В качестве примера дан расчёт решений для случая обтекания клина. Данный анализ может быть применен к течению ламинарного пограничного слоя с переменной скоростью свобoдного потока, когда значения критерия Прандтля несколько отличны от нуля.